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## A ratio-dependent predator-prey system model

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### 1 Introduction

The classical Lotka-Volterra model:

$$\dot{x} = ax - bxy \quad \dot{y} = -cy + dxy \quad (1)$$

where  $a, b, c$  and  $d$  are positive constants, has an extreme character such that all solutions are periodic and the average of each solution is equal to the equilibrium value,  $x = \frac{c}{d}$  and  $y = \frac{a}{b}$  [1]. However, once the saturation term is added as in the case of (2), there exists no non-constant periodic solution

$$\dot{x} = ax - bxy - x^2 \quad \dot{y} = -cy + dxy \quad (2)$$

(see [2]). This gap make the author doubt the validity of Lotka-Volterra type models.

On the other hand the author proposed a kind of ratio-dependent model for predator-prey system [3]. In this paper, first of all we shall show that our model possesses a non-constant periodic solution in spite of the appearance of saturation term and that the average of non-constant periodic solutions is less than the equilibrium value. Secondly we shall show that FitzHugh-Nagumo equation is a special case of our model, and hence FitzHugh-Nagumo equation is a kind of predator-prey system model. Thirdly we shall propose the model with time lag, which is reasonable from the aspect of biological theory and guarantees the positiveness of solutions.

### 2 Ratio-dependent model

The author proposed a kind of ratio-dependent model for prey and predator system such that

$$\frac{\dot{x}}{x} = a - \frac{by}{x} - g(x) \quad \frac{\dot{y}}{y} = -c + \frac{dx}{y} \quad (3)$$

where  $a, b, c$  and  $d$  are positive constants,  $x$  and  $y$  represent the populations of prey and predator,  $x > 0$  and  $y > 0$ , and  $g(x)$  represents the saturation effect, that is,  $g(x) > a$  for large  $x$  (see [3]). Obviously (3) is equivalent to that

$$\dot{x} = ax - by - g(x)x \quad \dot{y} = -cy + dx \quad (4)$$

We shall consider the existence of non-constant periodic solution of (4), which is positive valued. First of all we assume that the equation (5) has the positive root  $x^*$

$$g(x) = a - \frac{bd}{c}, \quad (5)$$

and hence  $E = (x^*, y^*)$ , where  $y^* = \frac{d}{c}x^*$ , is an equilibrium point.

### Theorem 1

Let  $g(x)$  be once continuously differentiable with respect to  $x > 0$ , and assume that  $g'(x^*) > 0$ ,  $g'(x^*)x^* = \frac{bd}{c} - c > 0$  and that  $\frac{\partial}{\partial a}g'(x^*)x^* \neq 0$ . Then there exists two continuously differentiable functions  $a(\varepsilon)$  and  $\omega(\varepsilon)$ ,  $a(0) = a$  and  $\omega(0) = \frac{\pi}{\sqrt{cg'(x^*)x^*}}$ , such that (4), where  $a = a(\varepsilon)$ , has a non-constant  $\omega(\varepsilon)$ -periodic solution  $(x(t, \varepsilon), y(t, \varepsilon))$  for  $\varepsilon \neq 0$  and  $(x(t, \varepsilon), y(t, \varepsilon)) \rightarrow E$  as  $\varepsilon \rightarrow 0$ . Consequently  $x(t, \varepsilon)$  and  $y(t, \varepsilon)$  are positive for small  $\varepsilon$ .

**Proof** The linear variational system of (4) around  $E$  is the following :

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} \frac{bd}{c} - g'(x^*)x^* & -b \\ d & -c \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix},$$

and hence the characteristic equation is

$$\lambda^2 + \left(g'(x^*)x^* - \frac{bd}{c} + c\right)\lambda + cg'(x^*)x^* = 0,$$

which, by our assumption, has the pure imaginary root  $\lambda = \pm 2i\sqrt{g'(x^*)x^*}$ . Since  $\frac{\partial}{\partial a}\{g'(x^*)x^* - \frac{bd}{c} + c\} \neq 0$ , our conclusion follows from Hopf bifurcation theorem [4, Theorem 4.1].

**Example 1** We shall treat the case where  $g(x) = x$ , and hence (4) is the following

$$\dot{x} = ax - by - x^2 \quad \dot{y} = -cy + dx \quad (6)$$

where  $bd > c^2$  and  $a = \frac{2bd}{c} - c$ . Then we may see that  $x^* = a - \frac{bd}{c} > 0$  and that  $g'(x^*)x^* - \frac{bd}{c} + c = a - \frac{2bd}{c} + c$ . Therefore we can verify

that all assumptions of Theorem 1 are satisfied, and consequently the conclusion of Theorem 1 holds for (6). Next let  $(x(t), y(t))$  be an existing non-constant periodic solution of (6) with period  $\omega > 0$ , and set  $x_0 = \frac{1}{\omega} \int_0^\omega x(t)dt$  and  $y_0 = \frac{1}{\omega} \int_0^\omega y(t)dt$ . From (6), we get that  $x_0 = \frac{c}{d}y_0$  and  $ax_0 = by_0 + \frac{1}{\omega} \int_0^\omega x^2(t)dt$ . Since  $\frac{1}{\omega} \int_0^\omega x^2(t)dt > x_0^2$ , it follows that  $(a - \frac{bd}{c})x_0 > x_0^2$ , which implies that  $x^* > x_0$ , and hence  $y^* > y_0$ . Namely the average of periodic solutions are smaller than the equilibrium values.

### 3 FitzHugh-Nagumo equation

We shall consider the case of (4) with external force  $(I, J)$ , that is,

$$\dot{x} = ax - by - g(x)x + I \quad \dot{y} = -cy + dx + J \quad (7)$$

Now we shall refer to the Bohnhoeffler-Van del Pol equation [5, p.447]

$$\dot{x} = c \left( y + x - \frac{x^3}{3} + z \right) \quad \dot{y} = -(x - a - by)/c$$

where  $a, b, c$  and  $z$  are constants. Replacing  $x$  by  $-x$ , we shall get

$$\dot{x} = cx - cy - \frac{c}{3}x^3 - cz \quad \dot{y} = \frac{1}{c}x - \frac{b}{c}y + \frac{a}{c},$$

which is the case of (7), where  $I = -cz$  and  $J = \frac{a}{c}$ . Next we shall refer to Nagumo's partial differential equation [6, p.2064]

$$\begin{aligned} h \frac{\partial^2 u}{\partial s^2} &= \frac{1}{c} \frac{\partial u}{\partial t} - w - \left( u - \frac{u^3}{3} \right) \\ c \frac{\partial w}{\partial t} + bw &= a - u, \end{aligned}$$

where  $a, b, c$  and  $h$  are constants. Replacing  $u$  by  $-x$  and  $w$  by  $y$  respectively, we shall get that

$$\begin{aligned} \frac{\partial x}{\partial t} &= ch \frac{\partial^2 x}{\partial s^2} + cx - cy - \frac{cx^3}{3} \\ \frac{\partial y}{\partial t} &= -\frac{b}{c}y + \frac{1}{c}x + \frac{a}{c}, \end{aligned}$$

which is the case of (7), where  $I = ch \frac{\partial^2 x}{\partial s^2}$  and  $J = \frac{a}{c}$ .

## 4 Delay system

The domain  $\{x \geq 0, y \geq 0\}$  may not be invariant for (4) as  $t$  increases. In order to cover this defect, we shall consider the case where (3) has partially a delay term such that

$$\frac{\dot{x}(t)}{x(t)} = a - b \frac{y(t-1)}{x(t-1)} - g(x(t)), \quad \frac{\dot{y}(t)}{y(t)} = -c + d \frac{x(t)}{y(t)} \quad (8)$$

where the initial condition is that  $x(\theta) > 0, y(\theta) > 0$  for  $-1 \leq \theta \leq 0$ . Let  $(x(t), y(t))$  denote the solution of (8).

### Theorem 2

$(x(t), y(t))$  is defined for  $t \geq 0$ ,  $x(t) > 0$  and  $y(t) > 0$  for  $t \geq 0$ , and  $(x(t), y(t))$  is bounded for  $t \geq 0$ .

**Proof** Setting that  $f(t) = a - b \frac{y(t-1)}{x(t-1)}$  for  $0 \leq t \leq 1$ , we shall obtain the ordinary differential equation such that

$$\dot{x}(t) = f(t)x(t) - g(x(t))x(t) \quad \dot{y} = -cy(t) + dx(t), \quad (9)$$

where  $0 \leq t \leq 1$ , and therefore by the usual existence theorem, (9) has the solution  $(x(t), y(t))$  for  $0 \leq t \leq 1$ . Repeating this argument infinitely, we may claim that the solution of (9) is defined for  $t \geq 0$ . Now the first equation of (9) yields that

$$x(t) = x(0) \exp \left( \int_0^t f(s) - g(x(s)) ds \right) > 0$$

and the second one that

$$y(t) = e^{-ct}y(0) + \int_0^t de^{-c(t-s)}x(s) ds > 0. \quad (10)$$

Since  $\dot{x}(t) < (a - g(x(t))x(t))$  and since there is a positive number  $A$  such that  $g(x) > a$  for  $x \geq A$ , it follows that  $x(t) < A$  for large  $t$ , and therefore (10) implies that  $y(t)$  is bounded for  $t \geq 0$ . The proof completed.

## References

1. Braum, M., Coleman, C.S., and Drew, D.A., edited (1983). Differential Equation Models, Springer-Verlag, New York.
2. Morita, Y.(1996). The Chaos of Biological Model (in Japanese), Asakura-shoten L.T.D., Japan.
3. Nakajima, F.(2004). Predator-prey system model of singular equations ; back to D'Ancona's queation, Hokkaido Univ. Preprint in Mathematics, No.635, Japan
4. Chow, S.N. and Hale, J.K.(1982). Methods of Bifurcation Theory, Chapt.1, Springer-Verlag, New York.
5. FitzHugh, R.(1961). Impulses and physiological states in theoretical models of nerve membrane, Biophysical J. Vol.1, 445-466.
6. Nagumo, J., Arimoto, s., and Yoshizawa, S.(1962). An active pulse transmission line simulating nerve axon, proceeding of the IRE, 2061-2070.

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